Introduction to the First Derivative

Activity 17

Most of the focus in the *Rates of Change* lab was on average rates of change. While the idea of rates of change at one specific instant was alluded to, we couldn't explore that idea formally because we hadn't yet talked about limits. Now that we have covered average rates of change and limits we can put those two ideas together to discuss rates of change at specific instances in time.

Suppose that an object is tossed into the air in such a way that the elevation of the object (measured in ft) is given by the function $s(t) = 40 + 40t - 16t^2$ where t is the amount of time that has passed since the object was tossed (measured in s). Let's determine the velocity of the object 2 seconds into this flight.

Recall that the difference quotient $\frac{s(2+h) - s(2)}{h}$ gives us the average velocity for the object between the times $t = 2$ and $t = 2 + h$. So long as h is positive, we can think of h as the length of the time interval. To infer the velocity exactly 2 seconds into the flight we need the time interval as close to 0 as possible; this is done using the appropriate limit in example 17.1.

Example 17.1

$$
\begin{aligned}
v(2) &= \lim_{h \to 0} \frac{s(2+h) - s(2)}{h} \\
&= \lim_{h \to 0} \frac{\left[40 + 40(2+h) - 16(2+h)^2\right] - \left[40 + 40(2) - 16(2)^2\right]}{h} \\
&= \lim_{h \to 0} \frac{40 + 80 + 40h - 64 - 64h - 16h^2 - 56}{h} \\
&= \lim_{h \to 0} \frac{-24h - 16h^2}{h} \\
&= \lim_{h \to 0} \frac{h(-24 - 16h)}{h} \\
&= \lim_{h \to 0} (-24 - 16h) \\
&= -24 - 16 \cdot 0 \\
&= -24
\end{aligned}
$$

From this we can infer that the velocity of the object 2 seconds into its flight is -24 ft/s. From that we know that the object is falling at a speed of 24 ft/s.

Problem 17.1

Suppose that an object is tossed into the air so that its elevation (measured in m) is given by the function $p(t) = 300 + 10t - 4.9t^2$ where t is the amount of time that has passed since the object was tossed (measured in s).

17.1.1 Evaluate $\lim_{h \to 0} \frac{p(4+h) - p(4)}{h}$ showing each step in the simplification process (as illustrated in example 17.1).

17.1.2 What is the unit for the value calculated in problem 17.1.1 and what does the value (including unit) tell you about the motion of the object?

17.1.3 Copy Table 17.1 onto your paper and compute and record the missing values. Do these values support your answer to problem 17.1.2?

Table 17.1: Average Velocities

t_1	$\frac{p(t_1) - p(4)}{t_1 - 4}$
3.9	
3.99	
3.999	
4.001	
4.01	
4.1	

Activity 18

In previous activities we saw that if p is a position function, then the difference quotient for p can be used to calculate average velocities and the expression $\lim_{h \to 0} \frac{p(t_0 + h) - p(t_0)}{h}$ calculates the instantaneous velocity at time t_0.

Graphically, the difference quotient of a function f can be used to calculate the slope of secant lines to f. What happens when we take the run of the secant line to zero? Basically, we are connecting two points on the line that are really, really, (***really***), close to one another. As mentioned above, sending h to zero turns an average velocity into an instantaneous velocity. Graphically, sending h to zero turns a secant line into ***a tangent line***.

The tangent line to the function $f(x) = \sqrt{x}$ at 4 is shown in Figure 18.1. A calculation of the slope of this line is shown in Example 18.1.

Example 18.1

$$
\begin{aligned}
m_{\tan} &= \lim_{h\to 0}\frac{f(4+h)-f(4)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{4+h}-\sqrt{4}}{h}\\
&= \lim_{h\to 0}\left(\frac{\sqrt{4+h}-2}{h}\cdot\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right)\\
&= \lim_{h\to 0}\frac{4+h-4}{h\left(\sqrt{4+h}+2\right)}\\
&= \lim_{h\to 0}\frac{h}{h\left(\sqrt{4+h}+2\right)}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}\\
&= \frac{1}{\sqrt{4+0}+2}\\
&= \frac{1}{4}
\end{aligned}
$$

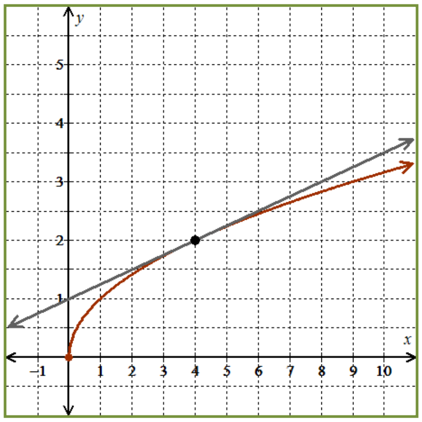

Figure 18.1: f

You should verify that the slope of the tangent line shown in Figure 18.1 is indeed $\frac{1}{4}$. You should also verify that the ***equation*** of the tangent line is $y=\frac{1}{4}x+1$.

Problem 18.1

Consider the function $g(x)=5-\sqrt{4-x}$.

18.1.1 Find the slope of the tangent line shown in Figure 18.2 using $m_{\tan}=\lim_{h\to 0}\frac{g(3+h)-g(3)}{h}$. Show work consistent with that illustrated in example 18.1.

18.1.2 Use the line in Figure 18.2 to verify your answer to problem 18.1.1.

18.1.3 State the equation of the tangent line to g at 3.

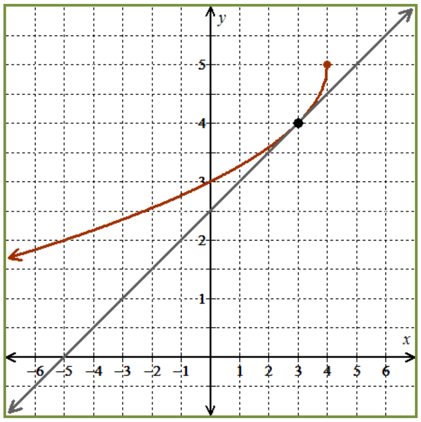

Figure 18.2: g

Activity 19

So far we've seen two applications of expressions of form $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$. It turns out that this expression is so important in mathematics, the sciences, economics, and many other fields that it deserves a name in and of its own right. We call the expression "***the first derivative of f at a***."

So far we've always fixed the value of a before making the calculation. There's no reason why we couldn't use a variable for a, make the calculation, and then replace the variable with specific values; in fact, it seems like this might be a better plan all around. This leads us to a definition of ***the first derivative function***.

Definition 19.1 – The First Derivative Function

If f is a function of x, then we define ***the first derivative function***, f', as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}.$$

The symbols $f'(x)$ are read aloud as "f prime of x" or "f prime at x."

As we've already seen, $f'(a)$ gives us the slope of the tangent line to f at a.

We've also seen that if s is a position function, then $s'(a)$ gives us the instantaneous velocity at a. It's not too much of a stretch to infer that the velocity function for s would be $v(t) = s'(t)$

Problem 19.1

A graph of the function $f(x) = \frac{3}{2-x}$ is shown in Figure 19.1 and the formula for $f'(x)$ is derived in Example 19.1.

19.1.1 Use the formula $f'(x) = \frac{3}{(2-x)^2}$ to calculate $f'(1)$ and $f'(5)$.

19.1.2 Draw onto Figure 19.1 a line through the point $(1,3)$ with a slope of $f'(1)$. Also draw a line though the point $(5,-1)$ with a slope of $f'(5)$. What are the names for the two lines you just drew? What are their equations?

19.1.3 Showing work consistent with that shown in example 19.1, find the formula for $g'(x)$ where $g(x) = \frac{5}{2x+1}$.

Example 19.1

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\dfrac{3}{2-(x+h)} - \dfrac{3}{2-x}}{h} \\
&= \lim_{h \to 0} \frac{\dfrac{3}{2-x-h} \cdot \dfrac{2-x}{2-x} - \dfrac{3}{2-x} \cdot \dfrac{2-x-h}{2-x-h}}{\dfrac{h}{1}} \\
&= \lim_{h \to 0} \left(\frac{6-3x-6+3x+3h}{(2-x-h)(2-x)} \cdot \frac{1}{h} \right) \\
&= \lim_{h \to 0} \frac{3h}{(2-x-h)(2-x)h} \\
&= \lim_{h \to 0} \frac{3}{(2-x-h)(2-x)} \\
&= \frac{3}{(2-x-0)(2-x)} \\
&= \frac{3}{(2-x)^2}
\end{aligned}
$$

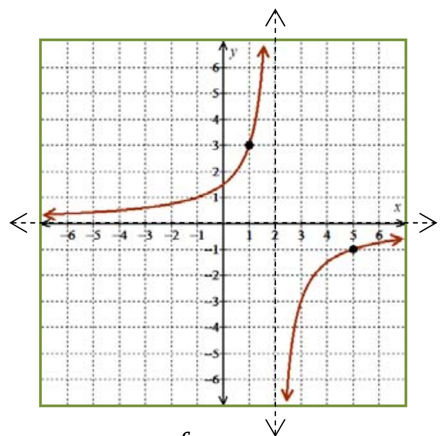

Figure 19.1: f

Problem 19.2

Suppose that the elevation of an object (measured in ft) is given by $s(t) = -16t^2 + 112t + 5$ where t is the amount of time that has passed since the object was launched into the air (measured in s).

19.2.1 Use Equation 19.2 to find the formula for the velocity function associated with this motion. The first two lines of your presentation should be an exact copy of Equation 19.2.

Equation 19.2

$$
\begin{aligned}
v(t) &= s'(t) \\
&= \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}
\end{aligned}
$$

19.2.2 Find the values of $v(2)$ and $v(5)$. What is the unit on each of these values? What do the values tell you about the motion of the object; don't just say "the velocity" – describe what is actually happening to the object 2 seconds and 5 seconds into its travel.

19.2.3 Use the velocity function to determine when the object reaches its maximum elevation. (Think about what must be true about the velocity at that instant.) Also, what is the common mathematical term for the point on the parabola $y = s(t)$ that occurs at that value of t?

19.2.4 Use Equation 19.3 to find the formula for $v'(t)$. The first line of your presentation should be an exact copy of Equation 19.3.

Equation 19.3 $$v'(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$$

19.2.5 What is the common name for the function $v'(t)$? Is its formula consistent with what you know about objects in freefall on Earth?

Problem 19.3

What is the constant slope of the function $w(x) = 12$? Verify this by using Definition 19.1 to find the formula for the function $w'(x)$.

Activity 20

We can think about the instantaneous velocity as being ***the instantaneous rate of change*** in position. In general, whenever you see the phrase "***rate of change***" you can assume that the rate of change at one instant is being discussed. When we want to discuss average rates of change over a time interval we always say "***average rate of change***."

In general, if *f* is any function, then **$f'(a)$ tells us the rate of change in f at a**. Additionally, if *f* is an applied function with an input unit of i_{unit} and an output unit of f_{unit}, then the unit on $f'(a)$ is $\frac{f_{unit}}{i_{unit}}$. Please note that this unit loses all meaning if it is simplified in any way. Consequently, *we do not simplify derivative units in any way, shape, or form.*

For example, if $v(t)$ is the velocity of your car (measured in mi/hr) where *t* is the amount of time that has passed since you hit the road (measured in minutes), then the unit on $v'(t)$ is $\frac{\text{mi/hr}}{\text{min}}$.

Problem 20.1

Determine the unit for the first derivative function for each of the following functions. Remember, *we do not simplify derivative units in any way, shape, or form.*

20.1.1 $V(r)$ is the volume of a sphere (measured in ml) with radius r (measured in mm).

20.1.2 $A(x)$ is the area of a square (measured in ft^2) with sides of length x (measured in ft).

20.1.3 $V(t)$ is the volume of water in a bathtub (measured in gal) where t is the amount of time that has elapsed since the tub began to drain (measured in minutes).

20.1.4 $R(t)$ is the rate at which a bathtub is draining (measured in gal/min) where t is the amount of time that has elapsed since the tub began to drain (measured in minutes).

Problem 20.2

Akbar was given a formula for the function described in problem 20.1.3. Akbar did some calculations and decided that the value of $V'(20)$ was (without unit) 1.5. Nguyen took one look and Akbar's value and said "that's wrong." What is it about Akbar's value that caused Nguyen to dismiss it as wrong?

Problem 20.3

After a while Nguyen convinced Akbar that he was wrong, so Akbar set about doing the calculation over again. This time Akbar came up with a value of $-12,528$. Nguyen took one look at Akbar's value and declared "still wrong." What's the problem now?

Problem 20.4

Consider the function described in problem 20.1.4.

20.4.1 What would it mean if the value of $R(t)$ was zero for all $t > 0$.

20.4.2 What would it mean if the value of $R'(t)$ was zero for all $0 < t < 2.25$.